Solution of the reduced anisotropic Maxwell-Bloch equations by using the Riemann-Hilbert problem

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We develop a method of solution for recently found integrable system of the reduced Maxwell-Bloch equations with two components of polarization and with an anisotropic dipole momentum by using the appropriate modification of the inverse scattering transform. The method is based on solution of the Riemann-Hilbert problem with taking into account symmetry properties of corresponding fundamental solutions. We show that these symmetries lead to some particular forms of the inverse scattering transform equations which may be used for finding as soliton-type as radiation-type solutions.

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I. INTRODUCTION

The generation and evolution of a few cycle optical pulses are of permanent interest because of their applications in various areas of physics, see, review [1]. The integrable reduced Maxwell-Bloch (RMB) equations [2] generalized in Refs. [3–7] have remarkable structural properties. For instance, the solutions of associated linear systems obey the nontrivial symmetry group. Analogous symmetry group had been revealed by Mikhailov [8] for the anisotropic Landau-Lifschitz equations, see also Ref. [9]. We will demonstrate that in a case of the integrable anisotropic RMB equations these symmetry properties require suitable modification of the inverse scattering transform technique.

Following the foregoing papers [3,4] we consider the interaction between an optical wave propagating in z-direction and an atomic two-level system with a dipole transition $\Delta J=0$, $\Delta M=1$ with J and M denoting the total angular momentum and its z-component, respectively. We assume an anisotropy insofar as two dipole moments $d_x \neq d_y$. Using the unidirectional approximation [2–4] for the Maxwell equations we found that the RMB equations take the form

$$(c\partial_z+\partial_t)\mathcal{E}_x=-2\pi d_x n \partial_t R_x,$$

$$(c\partial_z+\partial_t)\mathcal{E}_y=-2\pi d_y n \partial_t R_y, \tag{1}$$

$$\partial_t R_x=-\omega_0 R_y-\frac{2d_y}{\hbar}\mathcal{E}_y R_z, \quad \partial_t R_y=\omega_0 R_x+\frac{2d_x}{\hbar}\mathcal{E}_x R_z,$$

$$\partial_t R_z=\frac{2}{\hbar}(d_y\mathcal{E}_y R_x-d_x\mathcal{E}_x R_y). \tag{2}$$

Here $\mathcal{E}_x$ and $\mathcal{E}_y$ are the electric field components (R_x,R_y,R_z) is the Bloch vector, c is the velocity of light in the host medium, ω_0 is the resonance frequency, and n denotes the number density of atoms.

Denote

$$\chi=(4\pi d_x d_y n/\hbar c)z, \quad \tau_0=\omega_0(t-z/c), \tag{3}$$

$$E_{x,y}=(2\sqrt{d_x d_y}/\hbar\omega_0)\mathcal{E}_{x,y}, \tag{4}$$

$f=d_f/d_x$, where $(f\geqslant 1)$,

$$f_\pm=(f\pm f^{-1})/2, \quad h=f_-/(2f_+), \tag{5}$$

$$\theta=f_+\tau_0, \tag{6}$$

$$E=(E_x+iE_y)/\sqrt{f_+}, \quad R=(R_x\sqrt{f}+iR_y/\sqrt{f})/\sqrt{f_+}. \tag{7}$$

Then Eqs. (1) and (2) are

$$\partial_\chi E=-i(R+ER_z), \quad \partial_\chi F=i(S+FR_z),$$

$$\partial_\theta R_z=\frac{i}{2}(RF-SE), \quad \partial_\theta R=-\partial_\chi(E+2aF],$$

$$\partial_\theta S=-\partial_\chi(F+2aE), \tag{8}$$

where

$$F=\bar{E}, \quad S=\bar{R}, \quad R_z=\mathrm{Re}\,R_z. \tag{9}$$

The rest of the paper is organized as follows. In the next section the inverse scattering transform technique and the Riemann-Hilbert problem (RHP) formulation are presented. In Sec. III a soliton solution is found. In Sec. IV a symmetrized Fredholm equation solving the regular RHP is derived.

II. THE INVERSE SCATTERING TRANSFORM TECHNIQUE

System (8) possesses the following Lax pair [4],

$$\partial_\tau\Phi=\mathbf{U}\Phi:=\begin{pmatrix}-i(\lambda^2-\lambda^{-2}) & \mathcal{E}\lambda+\mathcal{F}/\lambda\\ -\mathcal{F}\lambda-\mathcal{E}/\lambda & i(\lambda^2-\lambda^{-2})\end{pmatrix}\Phi, \tag{10}$$

$$\partial_\chi\Phi=\mathbf{V}\Phi$$

$$:=-w_0(\lambda)\begin{pmatrix}-i\sqrt{h}(\lambda^2-\lambda^{-2})R_z & R\lambda+S/\lambda\\ -S\lambda-R/\lambda & i\sqrt{h}(\lambda^2-\lambda^{-2})R_z\end{pmatrix}\Phi, \tag{11}$$

where

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$$\mathcal{E}=\frac{E}{h}=\bar{\mathcal{F}},\quad \tau=\frac{1}{2}h\theta,\quad w_0(\lambda)=\frac{q}{2[1+h(\lambda^2+\lambda^{-2})]}. \tag{12}$$

Symmetry properties play a crucial role in developing the technique used here. Let us list the symmetries of the matrix functions $\mathbf{U}(\lambda)$, $\mathbf{V}(\lambda)$,

$$\mathbf{U}(1/\lambda)=\sigma_1\mathbf{U}(-\lambda)\sigma_1,\quad \mathbf{V}(1/\lambda)=\sigma_1\mathbf{V}(-\lambda)\sigma_1, \tag{13}$$

$$\mathbf{U}(-\lambda)=\sigma_3\mathbf{U}(\lambda)\sigma_3,\quad \mathbf{V}(-\lambda)=\sigma_3\mathbf{V}(\lambda)\sigma_3, \tag{14}$$

$$\overline{\mathbf{U}(-\bar{\lambda})}=\sigma_1\mathbf{U}(\lambda)\sigma_1,\quad \overline{\mathbf{V}(-\bar{\lambda})}=\sigma_1\mathbf{V}(\lambda)\sigma_1, \tag{15}$$

$$\overline{\mathbf{U}(1/\bar{\lambda})}=\mathbf{U}(\lambda),\quad \overline{\mathbf{V}(1/\bar{\lambda})}=\mathbf{V}(\lambda), \tag{16}$$

here σ_1 and σ_3 are standard Pauli matrices.

Define the group of transforms of a complex plane consisting in the identity transform I and in elements, acting as follows:

$$u_{g_1}(\lambda)=\frac{1}{\lambda},\quad u_{g_2}(\lambda)=-\lambda,\quad u_{g_3}(\lambda)=-\frac{1}{\lambda}. \tag{17}$$

Transforms $\{I,u_{g_1},u_{g_2},u_{g_3}\}$ forming an Abelian group $\mathcal{S}$ of substitutions include the parity transform u_{g_2}, the substitution u_{g_2} and the combined transform u_{g_3}, $g_3=g_1g_2$.

Define this group $\mathcal{G}$ as an automorphism group that acts on the set of fundamental solutions $\psi(\chi,\tau;\zeta)$ of Eqs. (10) and (11) in the following manner:

$$g:\psi(\chi,\tau;\zeta)\rightarrow\hat{U}(g)\psi(\chi,\tau;u_g(\zeta))\in\{\psi(\chi,\tau;\zeta)\}. \tag{18}$$

Group $\mathcal{G}$ also consists in the elements: $\{I,g_1,g_2,g_3\}$, $g_k=g_k^{-1}$, $g_i=g_jg_k$, $i\neq j\neq k$, acting as follows:

$$\hat{U}(g_1)\psi=\overline{\psi(\chi,\tau;u_{g_1}(\lambda))}, \tag{19}$$

$$\hat{U}(g_2)\psi=\sigma_3\psi(\chi,\tau;u_{g_2}(\lambda))\sigma_3, \tag{20}$$

$$\hat{U}(g_3)\psi=\sigma_3\overline{\psi(\chi,\tau;u_{g_3}(\lambda))}\sigma_3. \tag{21}$$

Taking into account symmetry properties (19)–(21) we find that transforms of the scattering coefficients $a(\chi;\lambda)$, $b(\chi;\lambda)$, see below Eq. (34), under action of elements of substitution group $\mathcal{S}$ are

$$a(\chi;u_{g_1}(\lambda))=\overline{a(\chi;\lambda)},\quad b(\chi;u_{g_1}(\lambda))=\overline{b(\chi;\lambda)}, \tag{22}$$

$$a(\chi;u_{g_2}(\lambda))=a(\chi;\lambda),\quad b(\chi;u_{g_2}(\lambda))=-b(\chi;\lambda), \tag{23}$$

$$a(\chi;u_{g_3}(\lambda))=\overline{a(\chi;\lambda)},\quad b(\chi;u_{g_3}(\lambda))=-\overline{b(\chi;\lambda)}. \tag{24}$$

Symmetry properties of the coefficient $c(\chi;\lambda_1)=b(\chi;\lambda_1)/\partial_\lambda a(\chi;\lambda)|_{\lambda=\lambda_1}$ where λ_1 is the simple zero of $a(\chi;\lambda)$ are the following:

$$c(\chi;u_{g_1}(\lambda_1))=-\frac{1}{\bar{\lambda}_1^2}\overline{c(\chi;\lambda_1)}, \tag{25}$$

$$c(\chi;u_{g_2}(\lambda_1))=c(\chi;\lambda_1), \tag{26}$$

$$c(\chi;u_{g_3}(\lambda_1))=-\frac{1}{\bar{\lambda}_1^2}\overline{c(\chi;\lambda_1)}. \tag{27}$$

Let all zeros λ_{0k}, $k=1,2,\ldots,n$, of $a(\chi;\lambda)$ are nondegenerate, i.e., $|\lambda_{0k}|\neq 1$ as well as $|\lambda_{0k}|\neq 0,\infty$. Symmetry properties (19)–(24) mean that poles λ_{0k}, $\lambda_{2k}=\overline{\lambda_{0k}}^{-1}$, $\lambda_{3k}=-\lambda_{0k}$, $\lambda_{4k}=-\overline{\lambda_{0k}}^{-1}$ are equivalent points in the complex plane, see below.

We consider here finite supported solutions decreasing in the infinities: $\mathcal{E}(\tau,\chi)\rightarrow 0$ as $\tau\rightarrow\pm\infty$. Pulses propagate over the trivial background

$$\mathcal{E}(\chi,\tau)\equiv 0. \tag{28}$$

The "boundary" conditions are

$$R_x(\chi,0)=\epsilon=\pm 1,\quad R(\chi,0)=S(\chi,0)=0. \tag{29}$$

We suppose the initial data of the Cauchy problem $\mathcal{E}(\tau,0)$, $\mathcal{F}(\tau,0)$, for Eq. (10) to be sufficiently smooth and to decrease sufficiently as $\tau\rightarrow\pm\infty$.

Introduce the matrix-valued functions,

$$\Phi'_-=(\phi',\tilde{\phi}'),\quad \Phi'_+=(\tilde{\psi}',\psi'), \tag{30}$$

here $\phi'=\phi'(\chi,\tau;\lambda)$, $\tilde{\phi}'=\tilde{\phi}'(\chi,\tau;\lambda),\ldots$ are the columns. Let these functions have an asymptotic behavior

$$\Phi'_\pm(\tau;\lambda)\rightarrow\exp[-i\Lambda(\lambda)\tau\sigma_3],\quad \tau\rightarrow\pm\infty, \tag{31}$$

here $\mathrm{Im}\,\lambda^2=0$, $\Lambda(\lambda)=\lambda^2-\lambda^{-2}$.

Let the Jost functions—fundamental solutions of (10)—possess the following forms:

$$\Phi^-=(e^{(-i\mu_-+i\mu_0)\sigma_3}\phi',e^{(i\mu_--i\mu_0)\sigma_3}\tilde{\phi}'):=(\phi,\tilde{\phi}),$$

$$\Phi^+=(e^{-i\mu_+\sigma_3}\tilde{\psi}',e^{i\mu_+\sigma_3}\psi'):=(\tilde{\psi},\psi), \tag{32}$$

here, μ_0 is a real function of χ and $\mu_\pm$ are the real functions of τ and χ such that

$$\lim_{\tau\rightarrow-\infty}\mu_-(\tau,\chi)=0,\quad \lim_{\tau\rightarrow\infty}\mu_+(\tau,\chi)=0. \tag{33}$$

$\mu_0(\chi)$, $\mu_\pm(\tau,\chi)$ do not depend on λ, see below.

The completeness relationship of the eigenfunctions is given by

$$\Phi^-=\Phi^+\mathbf{T},\quad \mathbf{T}=\begin{pmatrix} a(\lambda) & -\overline{b(\bar{\lambda})} \\ b(\lambda) & \overline{a(\bar{\lambda})}\end{pmatrix}, \tag{34}$$

where λ belong to contour $\Gamma=\{\lambda:\mathrm{Re}\,\lambda=0\cup\mathrm{Im}\,\lambda=0\}$, see Fig. 1. $\mathbf{T}$ is a scattering matrix.

Evolution of scattering data can be found in a standard manner by using linear system (11) for boundary conditions (29),

$$a(\chi;\lambda)=a(0;\lambda),\quad b(\chi;\lambda)=b(0;\lambda)\exp[2i\sqrt{h}\epsilon w_0(\lambda)\Lambda(\lambda)\chi], \tag{35}$$

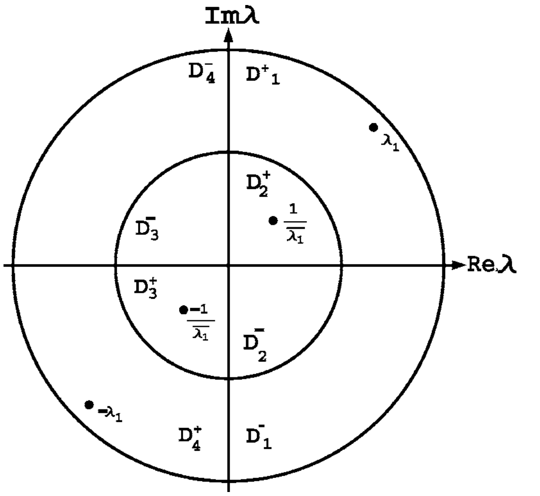

FIG. 1. The complex λ-plane. The inner circle has the unit radius and the outer circle has the radius $l_0 \to \infty$. Domains $D_k^{\pm}$ are placed between the intervals lying on the axis and the quarters of cycles. Contour Γ_k^- (Γ_k^+) runs along the boundary of domain D_k^- (D_k^+) in the clockwise (counterclockwise) direction. The equivalent poles positions are depicted by the bold points.

$$\lambda_{0n}(\chi) = \lambda_{0n}(0),$$

$$c(\chi;\lambda_{0n}) = c(0;\lambda_{0n})\exp[2i\sqrt{h}\epsilon w_0(\lambda_{0n})\Lambda(\lambda_{0n})\chi]. \quad (36)$$

Define the matrix functions,

$$\mathbf{M}(\tau;\lambda) \coloneqq (\phi e^{(i\mu_- - i\mu_0)\sigma_3 + i\Lambda\tau}, \tilde{\phi} e^{(-i\mu_- + i\mu_0)\sigma_3 - i\Lambda\tau}), \quad (37)$$

$$\mathbf{N}(\tau;\lambda) \coloneqq (\tilde{\psi} e^{-i\mu_+\sigma_3 + i\Lambda\tau}, \psi e^{i\mu_+\sigma_3 - i\Lambda\tau}), \quad (38)$$

having the asymptotics

$$\mathbf{M}(\tau;\lambda) = \mathbf{I}, \quad \tau \to -\infty, \quad \mathbf{N}(\tau;\lambda) = \mathbf{I}, \quad \tau \to \infty, \quad (39)$$

where **I** is the unite matrix.

Substitute expression (37) for $\mathbf{M}_1$ in system (10) and integrating resulting equations with taking into account the boundary conditions (39) yield

$$\mathbf{M}_1(\tau;\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{\infty} \mathbf{G}(\tau - s,\lambda)\mathbf{Q}(s;\lambda)\mathbf{M}_1(s;\lambda)ds, \quad (40)$$

where

$$\mathbf{Q}(\tau;\lambda) = \begin{pmatrix} i\partial_\tau \mu_- & (\lambda\mathcal{F} + \mathcal{E}\lambda^{-1})e^{2i\mu_- - 2i\mu_0} \\ -(\lambda\mathcal{F} + \mathcal{E}\lambda^{-1})e^{-2i\mu_- + 2i\mu_0} & -i\partial_\tau \mu_- \end{pmatrix}, \quad (41)$$

$$\mathbf{G}(\tau;\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\Lambda(\lambda)\tau} \end{pmatrix}\theta(\tau), \quad (42)$$

$\theta(\tau)$ is the theta function. Analogous equations may be found for $\mathbf{M}_2$, $\mathbf{N}_1$, $\mathbf{N}_1$.

Let the domains $D_j^{\pm}$, $j = 1, \ldots, 4$ which boundaries are depicted in Fig. 1, be defined by

$$D_1^+ = \{\text{Im}\,\lambda > 0 \cap \text{Re}\,\lambda > 0 \cap l_0 > |\lambda| > 1\},$$

$$D_1^- = \{\text{Im}\,\lambda < 0 \cap \text{Re}\,\lambda > 0 \cap l_0 > |\lambda| > 1\},$$

$$D_2^+ = \{\text{Im}\,\lambda > 0 \cap \text{Re}\,\lambda > 0 \cap |\lambda| < 1\},$$

$$D_2^- = \{\text{Im}\,\lambda < 0 \cap \text{Re}\,\lambda > 0 \cap |\lambda| < 1\},$$

$$D_3^+ = \{\text{Im}\,\lambda < 0 \cap \text{Re}\,\lambda < 0 \cap |\lambda| < 1\},$$

$$D_3^- = \{\text{Im}\,\lambda > 0 \cap \text{Re}\,\lambda < 0 \cap |\lambda| < 1\},$$

$$D_4^+ = \{\text{Im}\,\lambda < 0 \cap \text{Re}\,\lambda < 0 \cap l_0 > |\lambda| > 1\},$$

$$D_4^- = \{\text{Im}\,\lambda > 0 \cap \text{Re}\,\lambda < 0 \cap l_0 > |\lambda| > 1\},$$

where $l_0 \to \infty$.

Group $\mathcal{S}$ is the automorphism group of the regions of complex plane: $D^+ = D_1^+ \cup D_2^+ \cup D_3^+ \cup D_4^+$ and $D^- = D_1^- \cup D_2^- \cup D_3^- \cup D_4^-$. Therefore the standard fundamental domains are $D_1^+ = D^+/\mathcal{S}$ and $D_1^- = D^-/\mathcal{S}$, respectively. Points $\lambda_1^{\pm}$, $\lambda_2^{\pm} = \lambda_1^{\pm -1}$, $\lambda_3^{\pm} = -\lambda_1^{\pm}$, $\lambda_4^{\pm} = -\lambda_1^{\pm -1}$ are equivalent points, where $\lambda_1^{\pm} \in D_1^{\pm}$, respectively.

Restrict our consideration to λ lying in the fundamental domain D_1^+. In the limit $\Lambda \to \infty$ or $\lambda \to +\infty$ we obtain from Eqs. (40),

$$\mathbf{M}_1(\tau;\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{1}{2i\Lambda}\begin{pmatrix} P_-(\tau) \\ (\lambda\mathcal{F} + \mathcal{E}\lambda^{-1})e^{2i\mu_- - 2i\mu_0} \end{pmatrix} + O\left(\frac{1}{\Lambda^2}\right), \quad (43)$$

where

$$P_-(\tau) = \frac{1}{2}\int_{-\infty}^{\tau} [\mathcal{E}^2(\tau') + \mathcal{F}^2(\tau')]d\tau'. \quad (44)$$

Asymptotics (39) and symmetry condition (15) are valid if

$$\mu_{\pm}(\tau,\chi) = \frac{1}{2}\int_{\pm\infty}^{\tau} \mathcal{E}(\tau',\chi)\mathcal{F}(\tau',\chi)d\tau' \quad (45)$$

and

$$\mu_0(\chi) = \mu_-(\tau,\chi) - \mu_+(\tau,\chi) = \frac{1}{2}\int_{-\infty}^{\infty} \mathcal{E}(\tau',\chi)\mathcal{F}(\tau',\chi)d\tau'. \quad (46)$$

Decompositions analogous to (43) may be derived as for domain D_2^+ in the limit $\lambda \to +0$ as for domains D_3^+, $\lambda \to -0$ and D_4^+, $\lambda \to -\infty$. Resulting equations are equivalent to (43) due to symmetry conditions. Expressions for functions $\mu_{\pm}(\tau\chi), \mu_0(\chi)$ remain the same forms (45) and (46), respectively.

Define

$$\rho(\chi;\lambda)=\frac{b(\chi;\lambda)}{a(\chi;\lambda)},\quad \tilde{\rho}(\chi;\lambda)=\frac{\overline{b(\chi;\bar{\lambda})}}{\overline{a(\chi;\bar{\lambda})}} \tag{47}$$

and the matrix function,

$$\Psi_{+}(\tau,\chi;\lambda)=\left(\frac{\mathbf{M}_1(\tau,\chi;\lambda)}{a(\chi;\lambda)},\mathbf{N}_2\right),$$

$$\Psi_{-}(\tau,\chi;\lambda)=\left(\mathbf{N}_2,\frac{\overline{\mathbf{M}_1(\tau,\chi;\lambda)}}{\overline{a(\chi,\bar{\lambda})}}\right), \tag{48}$$

having the asymptotics

$$\lim_{\lambda\to\infty}\Psi_{\pm}(\tau,\chi;\lambda)=\mathbf{I}. \tag{49}$$

The matrix function Ψ_+ is analytical in region D^+ and Ψ_- is analytical in region D^-. A jump condition may be formulated for a pair of the fundamental domains D_1^+, D_1^- and a pair of domains D_1^+, D_4^- then the jump condition may be expanded to the whole complex plane. The jump conditions for functions $\Psi^+(\lambda)$ and $\Psi^-(\lambda)$ restricted to their domains of analyticity D_1^+, D_1^-, and D_1^+, D_4^- are, respectively,

$$\Psi_{+}(\tau,\chi;\lambda)=\Psi_{-}(\tau,\chi;\lambda)\mathbf{J}_{+}(\tau,\chi;\lambda),\quad \lambda\in D_1^+\cap D_1^-, \tag{50}$$

$$\Psi_{+}(\tau,\chi;\lambda)=\Psi_{-}(\tau,\chi;\lambda)\mathbf{J}_{-}(\tau,\chi;\lambda),\quad \lambda\in D_1^+\cap D_4^-. \tag{51}$$

The 2×2 matrices $\mathbf{J}_\pm$ are defined in terms of the spectral datum $\{a(\lambda),b(\lambda)\}$ by the following formulas:

$$\mathbf{J}_{\pm}(\tau,\chi;\lambda)=\begin{pmatrix}1\pm\rho(\chi;\lambda)\tilde{\rho}(\chi;\lambda) & \pm\tilde{\rho}(\chi;\lambda)e^{-2i\Lambda(\lambda)\tau}\\ \rho(\chi;\lambda)e^{2i\Lambda(\lambda)\tau} & 1\end{pmatrix}. \tag{52}$$

Domains $D_k^\pm$ and contours $\Gamma_k^\pm$ running along their respective boundaries, see Fig. 1, are mapped by group $\mathcal{S}$ transforms as follows:

$$u_{g_1}\{D_1^\pm\}=\{D_2^\pm\},\quad u_{g_2}\{D_1^\pm\}=\{D_3^\pm\},\quad u_{g_3}\{D_1^\pm\}=\{D_4^\pm\}, \tag{53}$$

$$u_{g_1}\{\Gamma_1^\pm\}=\{\Gamma_2^\pm\},\quad u_{g_2}\{\Gamma_1^\pm\}=\{\Gamma_3^\pm\},\quad u_{g_3}\{\Gamma_1^\pm\}=\{\Gamma_4^\pm\}. \tag{54}$$

Contours $\Gamma_k^\pm$ are mapped with changing the direction of integration.

Let $\mathbf{J}_{jk}$ be the jump matrix for $\lambda\in D_j^+\cap D_k^-$, then for the corresponding equivalent points $\lambda\in\Gamma$ we have

$$\mathbf{J}_{+}(\lambda)\equiv\mathbf{J}_{11}(\lambda)=\overline{\mathbf{J}_{22}(1/\bar{\lambda})}=\sigma_3\overline{\mathbf{J}_{33}(-1/\bar{\lambda})}\sigma_3=\sigma_3\mathbf{J}_{44}(-\lambda)\sigma_3,$$

$$\mathbf{J}_{-}(\lambda)\equiv\mathbf{J}_{14}(\lambda)=\overline{\mathbf{J}_{23}(1/\bar{\lambda})}=\sigma_3\overline{\mathbf{J}_{32}(-1/\bar{\lambda})}\sigma_3=\sigma_3\mathbf{J}_{41}(-\lambda)\sigma_3. \tag{55}$$

The RHP must be formulated for functions $\Psi^+(\lambda)$ and $\Psi^-(\lambda)$ analytical in regions D^+, D^- (except in a finite number of poles), respectively. Acting by operator $\hat{U}(g_k)$ on both sides of jump condition (50) and taking into account mapping (53) and (54), we obtain a jump condition on the boundary $D_k^+\cap D_k^-$ for functions Ψ^+, Ψ^- analytical in respective domains D_k^+, D_k^-. It is easily verified with taking into account relations (55) that owing to symmetry properties (19)–(24) the jump conditions appearing for each pair of corresponding domains D_j^+, D_k^-, $j,k=1,2,3,4$ and the boundaries between them have the form of Eqs. (50) and (51).

Then, the matrix-function Ψ for each fixed λ satisfies the jump condition which can be written in the common form

$$\Psi_{+}(\tau,\chi;\lambda)=\Psi_{-}(\tau,\chi;\lambda)\mathbf{J}_{+}(\tau,\chi;\lambda),\quad \mathrm{Im}\,\lambda=0, \tag{56}$$

$$\Psi_{+}(\tau,\chi;\lambda)=\Psi_{-}(\tau,\chi;\lambda)\mathbf{J}_{-}(\tau,\chi;\lambda),\quad \mathrm{Re}\,\lambda=0, \tag{57}$$

where Ψ is Ψ_+ for $\lambda\in D^+$, Ψ is Ψ_- for $\lambda\in D^-$.

Equations (56) and (57) combined with (49) is known in the literature [10] as the Riemann-Hilbert problem with canonical normalization.

Consider functions $\Psi^\pm(\lambda)$ restricted to the corresponded fundamental domains: $\lambda\in D_1^\pm$. For $a(\lambda)\neq 0$, applying projections on the first columns to Eqs. (56) and (57) present the solution of the RHP in the following form:

$$\Psi_1^-(\chi,\tau;\lambda)=\begin{pmatrix}1\\0\end{pmatrix}+P_\lambda\hat{G}\frac{1}{2\pi i}\int_{\Gamma_1^+}\rho(\chi;\zeta)e^{2i\Lambda(\zeta)\tau}\times\Psi_2^+(\chi,\tau;\zeta)\frac{d\zeta}{\zeta-\lambda}, \tag{58}$$

where $\Psi_k^\pm$ is the kth column of matrix function $\Psi^\pm$. The following symmetry properties make system of equations closed:

$$\Psi_{12}^+(\chi,\tau;\lambda)=-\overline{\Psi_{21}^-(\chi,\tau;\bar{\lambda})},\quad \Psi_{22}^+(\chi,\tau;\lambda)=\overline{\Psi_{11}^-(\chi,\tau;\bar{\lambda})}. \tag{59}$$

The symmetrization operator

$$\hat{G}=(1+\hat{U}_{g_1})(1+\hat{U}_{g_2}) \tag{60}$$

is introduced on the right-hand side of Eq. (58) to satisfy the symmetry properties of the fundamental solution. The projector P_λ acts as follows:

$$P_\lambda\Phi(\lambda)=\Phi(\lambda)-\lim_{\lambda\to\infty}\Phi(\lambda) \tag{61}$$

is introduced to satisfy the canonical normalization (49). Projector P_λ obeys the symmetry conditions

$$\hat{U}_{g_k}P_\lambda=P_{u_{g_k}(\lambda)}. \tag{62}$$

Using symmetry properties (19)–(21) and (62) one can easily find that the analogous limit takes place for $\lambda'\to 0$. In this limit we must take into account that $|\lambda|>1$ in Eq. (71) and use transform $\lambda'=1/\bar{\lambda}$.

Rewrite integrals on the right-hand side of Eq. (58) in the explicit form

$$\hat{G}\frac{1}{2\pi i}\int_{\Gamma_1^+}\rho(\chi,\zeta)e^{2i\Lambda(\zeta)\tau}\Psi_2^+(\chi,\tau;\zeta)\frac{d\zeta}{\zeta-\lambda}$$
$$=\frac{1}{2\pi i}\Bigg[\int_{\Gamma_1^+}\frac{\rho(\chi,\zeta)e^{2i\Lambda(\zeta)\tau}\Psi_2^+(\chi,\tau;\zeta)d\zeta}{\zeta-\lambda} - \int_{\Gamma_1^+}\overline{\left(\frac{\rho(\chi,\zeta)e^{2i\Lambda(\zeta)\tau}\Psi_2^+(\chi,\tau;\zeta)d\zeta}{\zeta-\overline{\lambda^{-1}}}\right)} - \int_{\Gamma_1^+}\overline{\left(\frac{\rho(\chi,\zeta)e^{2i\Lambda(\zeta)\tau}\sigma_3\Psi_2^+(\chi,\tau;\zeta)d\zeta}{\zeta+\overline{\lambda^{-1}}}\right)} + \int_{\Gamma_1^+}\frac{\rho(\chi,\zeta)e^{2i\Lambda(\zeta)\tau}\sigma_3\Psi_2^+(\chi,\tau;\zeta)d\zeta}{\zeta+\lambda}\Bigg]. \tag{63}$$

Next we extend the Cauchy integrals with integration along contour Γ_1^+ in the above expression (63) to integration along contours over all domains of region D^+ using symmetry properties of the fundamental solutions. Group $\mathcal{G}$ transformation operators $\hat{U}(g_k)$ include transforms of substitution group u_{g_k} and transforms t_{g_k}, see (19)–(21), acting as follows:

$$t_{g_1}[\mathbf{V}]=\bar{\mathbf{V}},\quad t_{g_1}[\nu]=\bar{\nu}, \tag{64}$$

$$t_{g_2}[\mathbf{V}]=\sigma_3\mathbf{V},\quad t_{g_2}[\nu]=\nu, \tag{65}$$

$$t_{g_3}[\mathbf{V}]=\sigma_3\bar{\mathbf{V}},\quad t_{g_3}[\nu]=\bar{\nu}, \tag{66}$$

here $\mathbf{V}$ is an arbitrary complex valued two-component vector and ν is a scalar.

Consider the Cauchy integral

$$C(\lambda)=\frac{1}{2\pi i}\int_{\Gamma_1^+}\frac{\mathbf{Q}(\zeta)d\zeta}{\zeta-\lambda}, \tag{67}$$

here a vector-function $\mathbf{Q}(\zeta)$ obeys the symmetry properties

$$\mathbf{Q}(u_{g_k}(\zeta))=t_{g_k}[\mathbf{Q}(\zeta)]. \tag{68}$$

Using property (68) rewrite the transformed Cauchy integral $\hat{U}(g_k)C(\lambda)$ in the following way:

$$\hat{U}(g_k)C(\lambda)=t_{g_k}\{C[u_{g_k}(\lambda)]\}$$
$$=\frac{1}{2\pi t_{g_k}[i]}\int_{\Gamma_1^+}\frac{t_{g_k}[\mathbf{Q}(\zeta)]dt_{g_k}[\zeta]}{t_{g_k}[\zeta]-t_{g_k}[u_{g_k}(\lambda)]}$$
$$=\frac{1}{2\pi t_{g_k}[i]}\int_{\Gamma_1^+}\frac{\mathbf{Q}(u_{g_k}(\zeta))dt_{g_k}[\zeta]}{t_{g_k}[\zeta]-t_{g_k}[u_{g_k}(\lambda)]}$$
$$=t_{g_k}\left(\frac{1}{2\pi i}\int_{\Gamma_k^+}\frac{t_{g_k}[\mathbf{Q}(\xi)]du_{g_k}^{-1}(\xi)}{u_{g_k}^{-1}(\xi)-u_{g_k}(\lambda)}\right). \tag{69}$$

In the last integral in Eq. (69) we change the integration variable ζ to $\xi=u_{g_k}(\zeta)$. It corresponds to mapping of domains $u_{g_1}\{D_1^+\}=\{D_k^+\}$ (53) and mapping of related contours $u_{g_1}\{\Gamma_1^+\}=\{\Gamma_k^+\}$ (54) with changing the direction of integration.

Introduce the functions

$$\mathbf{Q}(\chi,\tau;\zeta)=\rho(\chi;\zeta)\Psi_2^+(\chi,\tau;\zeta)e^{2i\Lambda(\zeta)\tau},$$

$$\tilde{\mathbf{Q}}(\chi,\tau;\zeta)=\tilde{\rho}(\chi;\zeta)\Psi_1^-(\chi,\tau;\zeta)e^{-2i\Lambda(\zeta)\tau} \tag{70}$$

obeying the property (68). Then, repeating the procedure of the extension of integrals regions presented in (63) and applying procedure of integral transform (69) to the integrals on the right-hand side of (63) we rewrite Eq. (58) in the following forms:

$$\Psi_1^-(\chi,\tau;\lambda)=\begin{pmatrix}1\\0\end{pmatrix}+\frac{1}{2\pi i}P_\lambda\Bigg(\int_{\Gamma_1^+}\frac{\mathbf{Q}(\chi,\tau;\zeta)d\zeta}{\zeta-\lambda} + \int_{\Gamma_2^+}\frac{\mathbf{Q}(\chi,\tau;\zeta)d\zeta}{\zeta^2(\zeta^{-1}-\lambda^{-1})}+\int_{\Gamma_3^+}\frac{\mathbf{Q}(\chi,\tau;\zeta)d\zeta}{\zeta-\lambda} + \int_{\Gamma_4^+}\frac{\mathbf{Q}(\chi,\tau;\zeta)d\zeta}{\zeta^2(\zeta^{-1}-\lambda^{-1})}\Bigg). \tag{71}$$

Integral equation (71) with condition (59) solve the RHP (56) and (57) with canonical normalization (49) and solutions are automorphic.

Taking limit $\lambda\to\infty$ in Eqs. (43) and (71) we find the following relation for coefficients before $1/\lambda$:

$$\mathcal{F}(\chi,\tau)e^{-2i\mu_+(\chi,\tau)}=-\frac{2}{\pi}\int_{\Gamma_1^++\Gamma_2^+}\rho(\chi,\lambda)\Psi_{22}^+(\chi,\tau;\lambda)e^{2i\Lambda(\lambda)\tau}d\lambda,\quad \lambda\to\infty. \tag{72}$$

III. THE SIMPLEST NONDEGENERATE SOLUTION

Let us find a solution corresponding to one pole ζ_1 lying inside the fundamental domain D_1^+. We do not take into account the contribution of the continuous spectrum. This solution corresponds to the boundary condition $R_z(\chi,0)=-1$.

Residue calculated in a pole $\zeta_k\in D_1^+$ in the second integral on the right-hand side of equality (63) is equal to residue calculated in the equivalent pole $\xi_k=1/\bar{\zeta}_k\in D_2^+$ in the second integral on the right-hand side of Eq. (71) and so on. Using this fact and denoting $\chi_1(\zeta)=\Psi_{12}^+(\chi,\tau;\zeta)$, $\chi_2(\zeta)=\Psi_{22}^+(\chi,\tau;\zeta)$ and making residues in ζ_1, we derive from system (58),

$$\begin{pmatrix}\overline{\chi_2(\bar{\lambda})}\\ -\overline{\chi_1(\bar{\lambda})}\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}+\frac{c_1(\chi)e^{2i\Lambda(\zeta_1)\tau}}{\zeta_1-\lambda}\begin{pmatrix}\chi_1(\zeta_1)\\ \chi_2(\zeta_1)\end{pmatrix} + \overline{c_1(\chi)e^{2i\Lambda(\zeta)\tau}}\left(\frac{1}{\overline{\zeta_1}-\lambda^{-1}}-\frac{1}{\overline{\zeta_1}}\right)\begin{pmatrix}\overline{\chi_1(\zeta_1)}\\ \overline{\chi_2(\zeta_1)}\end{pmatrix} + \frac{\sigma_3c_1(\chi)e^{2i\Lambda(\zeta_1)\tau}}{\zeta_1+\lambda}\begin{pmatrix}\chi_1(\zeta_1)\\ \chi_2(\zeta_1)\end{pmatrix} + \sigma_3\overline{c_1(\chi)e^{2i\Lambda(\zeta)\tau}}\left(\frac{1}{\overline{\zeta_1}+\lambda^{-1}}-\frac{1}{\overline{\zeta_1}}\right)\begin{pmatrix}\overline{\chi_1(\zeta_1)}\\ \overline{\chi_2(\zeta_1)}\end{pmatrix}, \tag{73}$$

where $c_1(\chi)=b(\chi;\zeta_1)/\partial_\zeta a(\chi;\zeta)|_{\zeta=\zeta_1}$. For solution of system (73) it is enough to set $\lambda=\bar{\zeta}_1$. Alternatively, one can use $\lambda=-\bar{\zeta}_1$ or ζ_1^{-1}, or $-\zeta_1^{-1}$. Using symmetry properties (19)–(21) one obtains the same systems of algebraic equations.

The simplest nondegenerate solution corresponding to one poles ζ_1 lying in the fundamental domain D_1^+ is found by using (72),

$$\mathcal{F}(\chi,\tau) = -\frac{4e^{2i\mu_+(\chi,\tau)+\vartheta(\chi,\tau)}[1-\bar{A}(\chi,\tau)-B(\chi,\tau)]}{1-A(\chi,\tau)-\bar{A}(\chi,\tau)+A(\chi,\tau)\bar{A}(\chi,\tau)-B(\chi,\tau)\bar{B}(\chi,\tau)} - \frac{(\text{c.c.})}{\bar{\zeta}_1^2} \quad (74)$$

where

$$2\mu_+(\chi,\tau)=\int_{+\infty}^{\tau}|\mathcal{F}(t',\chi)|^2dt',$$

$$\vartheta(\chi,\tau)=-i\Lambda(\zeta_1)[\tau+2qw_0(\zeta_1)\chi]+\ln|c_1(0)|+i\arg\{c_1(0)\},$$

$$\bar{A}(\chi,\tau)=-f_1(\chi,\tau)\frac{8|\zeta_1|^2e^{i\phi_1}}{q_1p_1}\times\cosh[i\,\text{Im}\,\vartheta(\chi,\tau)+2i\arg\zeta_1+\ln|\zeta_1|].$$

$$B(\chi,\tau)=-if_1(\chi,\tau)\frac{8|\zeta_1|^2e^{i\arg\zeta_1}}{q_1p_1}\times\cosh\left(i\,\text{Im}\,\vartheta(\chi,\tau)+2i\arg\zeta_1+i\phi_1-\ln\frac{p_1}{q_1}\right),$$

$$f_1(\chi,\tau)=\exp[2\vartheta(\chi,\tau)-3i\,\text{Im}\,\vartheta(\chi,\tau)],$$

$$q_1=2\,\text{Im}\,\zeta_1^2,\quad p_1=|\bar{\zeta}_1(\bar{\zeta}_1^4-1)|,\quad \phi_1=\arg[\bar{\zeta}_1(\bar{\zeta}_1^4-1)].$$

Using $|\zeta_1|>1$ and $\text{Im}\,\zeta_1^2>0$ one can prove that solution (74) is nonsingular, i.e., denominator of (74) is positive for any χ and τ.

The starting table of symbols, used in Eqs. (10) and (11), shows that anisotropy contribution is determined by parameter h. In such a table of symbols domain $D_2=D_2^+\cap D_2^-$ lies within a circle having the radius h and a center at point $\lambda=0$, see Fig. 1. If $h\to 0$ then this domain vanishes and term B disappears in above solution. This physical limit corresponds to propagation of the circularly polarized light in an isotropic medium. Solution (74) transforms into soliton having structure analogous to that of the nonlinear differential Shrödinger equation [11] up to χ-dependence of function ϑ,

$$\mathcal{F}(\chi,\tau)=-\frac{4e^{\vartheta(\chi,\tau)}}{1-A_1(\chi,\tau)}\exp\left(i\int_{\infty}^{\tau}\left|\frac{4e^{\vartheta(\chi,t')}}{1-A_1(\chi,t')}\right|^2dt'\right), \quad (75)$$

where

$$A_1(\chi,\tau)=\left(\frac{2\zeta_1}{\zeta_1^2-\bar{\zeta}_1^2}\right)^2e^{2\,\text{Re}\,\vartheta(\chi,\tau)}.$$

In a case of degenerate zeros, $|\zeta_1|=1$, $\arg\zeta_1\neq 0$, $\pi/2$ solution (74) has a sech-type form as well.

IV. FREDHOLM EQUATION

In this section we derive the symmetrized Fredholm equation. Define the symmetrized Cauchy-type integrals,

$$\tilde{C}(\lambda)=P_\lambda\hat{G}\frac{1}{2\pi i}\int_{\Gamma'}\frac{f(\zeta)d\zeta}{\zeta-\lambda}, \quad (76)$$

having zero asymptotic as $\lambda\to\infty$, where $\Gamma'=\Gamma_\pm,\Gamma_4$.

If a density $f(\lambda)$ satisfies the Hölder condition, the function $\tilde{C}(\lambda)$ has the jumps on $D_1^+\cap D_1^-$ and on $D_1^+\cap D_4^-$ [10]

$$\tilde{C}_+(\lambda)-\tilde{C}_-(\lambda)=f(\lambda),\quad \lambda\in(D_1^+\cap D_1^-)\cup(D_1^+\cap D_1^-). \quad (77)$$

We suppose that $\tilde{C}(\lambda),\tilde{C}_\pm(\lambda)$ obey the symmetry properties (19)–(21). As a consequence jump condition (77) may be extended to whole λ-plane in the same manner as used above

$$\tilde{C}_+(\lambda)-\tilde{C}_-(\lambda)=f(\lambda),\quad \lambda\in\Gamma. \quad (78)$$

Then $f(\lambda)$ is a boundary value of some function analytical in regions D^+ and D^- bounded by the contour Γ.

Consider a solitonless sector. Let $\Psi_\pm$ be solution of the jump condition (56), then the integral Cauchy formulas are valid,

$$\Psi_-(\lambda)=-P_\lambda\hat{G}\frac{1}{2\pi i}\int_{\Gamma_1^-}\frac{\Psi_-(\zeta)d\zeta}{\zeta-\lambda},\quad \lambda\in D^-, \quad (79)$$

$$\Psi_+(\lambda)=P_\lambda\hat{G}\frac{1}{2\pi i}\int_{\Gamma_1^+}\frac{\Psi_+(\zeta)d\zeta}{\zeta-\lambda}+\Psi_0,\quad \lambda\in D^+, \quad (80)$$

here Ψ_0 is some constant normalization matrix normalized as $\lambda\to\infty$, see below.

On the contour Γ, using (56), we obtain

$$\frac{1}{2}\Psi_-(\lambda)=-P_\lambda\hat{G}\frac{1}{2\pi i}\int_{D_1^+\cap D_1^-\cup D_1^+\cap D_4^-}\frac{\Psi_-(\zeta)d\zeta}{\zeta-\lambda}, \quad (81)$$

$$\frac{1}{2}\Psi_-(\lambda)\mathbf{J}(\lambda)=P_\lambda\hat{G}\frac{1}{2\pi i}\int_{D_1^+\cap D_1^-\cup D_1^+\cap D_4^-}\frac{\Psi_-(\zeta)\mathbf{J}(\zeta)d\zeta}{\zeta-\lambda}+\Psi_0, \quad (82)$$

here and below $\mathbf{J}(\lambda)=\mathbf{J}_+(\lambda)$ if $\text{Im}\,\lambda=0$ and $\mathbf{J}(\lambda)=\mathbf{J}_-(\lambda)$ if $\text{Re}\,\lambda=0$.

Multiplying Eq. (81) on $J(\lambda)$, adding Eqs. (81) and (82) we obtain the Fredholm integral equation in operator form,

$$\Psi_-(\lambda)\mathbf{J}(\lambda)=P_\lambda\hat{G}\frac{1}{2\pi i}\int_{D_1^+\cap D_1^-\cup D_1^+\cap D_4^-}\frac{\Psi_-(\zeta)[\mathbf{J}(\zeta)-\mathbf{J}(\lambda)]d\zeta}{\zeta-\lambda}+\Psi_0. \quad (83)$$

Using symmetry properties (19) and (20) we rewrite the Fredholm equation

$$\Psi_{-}(\lambda)\mathbf{J}(\lambda) = \frac{1}{2\pi i}P_{\lambda}\left(\int_{(D_1^+\cap D_1^-\cup D_1^+\cap D_4^-)\cup(D_4^+\cap D_4^-\cup D_1^-\cap D_4^+)}\frac{\Psi_{-}(\zeta)[\mathbf{J}(\zeta)-\mathbf{J}(\lambda)]d\zeta}{\zeta-\lambda} + \int_{(D_2^+\cap D_2^-\cup D_2^+\cap D_3^-)\cup(D_3^+\cap D_3^-\cup D_2^-\cap D_3^+)}\frac{\Psi_{-}(\zeta)[\mathbf{J}(\zeta)-\mathbf{J}(\lambda)]d\zeta}{\zeta^2(\zeta^{-1}-\lambda^{-1})}\right) + \Psi_0. \tag{84}$$

Symmetry with respect to action of elements g_k of group $\mathcal{G}$ imposes a set of restrictions to matrix Ψ_0. Invariance under symmetry transform g_1 makes this matrix real and the parity conservation related with transform g_2 makes the matrix Ψ_0 diagonal. Equation (84) may be used to find a radiation solution corresponding to the regular RHP.

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